Math 246B Lecture 5 Notes

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1 More Properties of Subharmonic Functions

1.1 Uniqueness of subharmonic functions

Definition 1.1. Denote $SH(\Omega)$ to be the set of all subharmonic functions in Ω.

Last time, we showed that if $u \in SH(\Omega)$ and if $\{|x-a| \leq R\} \subseteq \Omega$, then

$$u(x) \leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a,y) u(a-y)\, ds(y), \qquad |x-a| < R.$$

Now assume that u is upper semicontinuous in $\{|x-a| \leq R\}$ and subharmonic in $\{|x-a| < R\}$. Then

$$u(x) \leq \frac{1}{2\pi r} \int_{|y|=r} P_r(x-a,y) u(a+y)\, ds(y), \qquad |x-a| < R.$$

To let $r \to R$, we can assume that $u \leq 0$ and apply Fatou's lemma. So

$$\begin{aligned} u(x) &\leq \limsup_{r \to R} \frac{1}{2\pi r} \int_{|y|=r} P_r(x-a,y) u(a+y)\, ds(y) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \limsup_{r \to R} \frac{r^2 - |x-a|^2}{|re^{it} - (x-a)|^2} u(a + re^{it})\, dt \\ &\leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a,y) u(a+y)\, ds(y). \end{aligned}$$

Proposition 1.1. *Let $f \in C(|z| \leq R) \cap \mathrm{Hol}(|z| < R)$. Assume that there exists a Lebesgue measurable $E \subseteq \{|z| = R\}$ of positive measure such that $f|_E = 0$. Then $f \equiv 0$ in $|z| < R$.*

Proof. We may assume that $|f| \leq 1$. The function $u = \log |f|$ is upper semicontinuous on $|z| = R$, subharmonic in $|z| < R$, so by our previous discussion,

$$\log |f(z)| \leq \frac{1}{2\pi R} \int_{|w|=R} \frac{R^2 - |z|^2}{|z-w|^2} \log |f(w)|\, |dw|, \qquad |z| < R.$$

The integrand equals $-\infty$ on E with $m(E) > 0$, so $f \equiv 0$. $\square$

1.2 Local integrability of subharmonic functions

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open and connected, and let $u \in SH(\Omega)$ with $u \not\equiv -\infty$. Then $u \in L^1_{\mathrm{loc}}(\Omega)$; that is, if $K \subseteq \Omega$ is compact, then $\int_K u(x)\, dx > -\infty$. Furthermore, if $\{|x-a| \leq R\} \subseteq \Omega$, then $\int_{|x-a|=R} u(x)\, ds(x) > -\infty$.*

Remark 1.1. The set $\{x \in \Omega : u(x) = -\infty\}$ is a Lebesgue-null set.

Proof. Let E be the set of points $x \in \Omega$ having a neighborhood ω such that $\overline{\omega} \subseteq \Omega$ and $\int_\omega u(x)\, dx > -\infty$. $E \neq \varnothing$ because there exists some $a \in \Omega$ with $u(a) > -\infty$, and the sub-mean value inequality gives

$$u(a) \leq \frac{1}{\pi R^2} \iint_{|x-a|<R} U(x)\, dx$$

for all small $R > 0$. E is also open.

Let us show that $\Omega \setminus E$ is open. If $\Omega \setminus E$ is not open, then there exists $a \in \Omega \setminus E$ and a sequence $a_n \in E$ such that $a_n \to a$. Arbitrarily close to a_n, there exists b_n such that $u(b_n) > \infty$. Picking b_n so that $|a_n - b_n| \to 0$, we get $b_n \to a$ and $u(b_n) > -\infty$ for all n. Take $R > 0$ such that $\{|x-a| < R \subseteq \Omega\}$. Then if $K_n = \{|x - b_n| \leq R/2\}$, we have $K_n \subseteq \Omega$ for large n. So

$$\frac{1}{\pi (R/2)^2} \iint_{K_n} u(x)\, dx \geq u(b_n) > -\infty.$$

For large n, $a \in K_n^o$. So $a \in E$, which contradicts the choice of a. Because Ω is connected, it follows that $\Omega = E$, and therefore $u \in L^1_{\mathrm{loc}}(\Omega)$.

If $\{|x-a| \leq R\} \subseteq \Omega$, write

$$u(x) \leq \frac{1}{2\pi R} \int_{|y|=R} P_r(x-a, y) u(a+y)\, ds(y), \qquad |x-a| < R.$$

We may assume that $u \leq 0$, and then

$$P_R(x-a, y) = \frac{R^2 - |x-a|^2}{|y-(x-a)|^2} \geq \frac{R^2 - \rho^2}{(R+\rho)^2} = \frac{R-\rho}{R+\rho}, \qquad \rho = |x-a|,$$

so

$$u(x) \leq \frac{1}{2\pi R} \frac{R-\rho}{R+\rho} \int_{|y|=R} u(a+y)\, ds(y).$$

This integral must be finite, for otherwise, $u = \infty$ on $|x-a| < R$. $\square$

1.3 Differential characterization of subharmonic functions

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u \in C^2(\Omega, \mathbb{R})$. Then $u \in SH(\Omega)$ if and only if $\Delta u \geq 0$ in Ω.*

Proof. ($\Longrightarrow$): Taylor expand u at $a \in \Omega$:

$$u(x) = u(a) + u'(a)(x-a) + \frac{1}{2}u''(a)(x-a)(x-a) + o(|x-a|^2),$$

where $u'(a) = (u'_{x_1}(a), u'_{x_2}(a))$ and $u''(a) = (u''_{x_j x_k}(a))_{1 \le j,k \le 2}$. Because u is subharmonic, for all small $R > 0$,

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})\, dt.$$

Substituting in the Taylor expansion, the linear terms drop out, and $(x_j - a_j)(x_k - a_k)$ drop out as well, when $j \neq k$. The remaining terms are the diagonal terms, which are exactly given by the Laplacian. So

$$u(a) \le u(a) + \frac{R^2}{4}\Delta u(a) + o(R^2).$$

We get

$$\frac{R^2}{4}\Delta u(a) + o(R^2) \implies \Delta u(a) \ge 0.$$

($\Longleftarrow$): Assume first that $\Delta u > 0$ in Ω. By the previous computation,

$$\frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})\, dt = u(a) + \frac{R^2}{4} \underbrace{\Delta u(a)}_{>0} + o(R^2) > u(a).$$

for small $R > 0$. Thus, $\Delta u > 0 \implies u \in SH(\Omega)$. In general, consider $u_\varepsilon = u + \varepsilon |x|^2$ for $\varepsilon > 0$. Then $\Delta u_\varepsilon \ge 4\varepsilon > 0$, so $u_\varepsilon \in SH(\Omega)$. Letting $\varepsilon \downarrow 0$, we get $u = \lim u_\varepsilon \in SH(\Omega)$. $\square$